NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-770

C^1—Compatible Interpolation Over a Triangle

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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Preface

The work described herein was performed by the Data Systems Division of the Jet Propulsion Laboratory.

Contents

1.	Introduction	1
2.	The problem of C^1 - compatible interpolation over a triangle	2
3.	Some observations on one-dimensional Hermite cubic interpolation	4
4.	Coordinate systems over plane triangles	9
5.	Constructing a solution	15
6.	Correcting the normal derivatives of $g_1=r_2r_3(r_2-r_3)$	19
7.	A set of rational correction functions, ρ_i	22
8.	A set of piecewise cubic correction functions, ρ_i	24
9.	Transforming formulas to algorithms	27
10.	The method described in Goël [1968]	31
11.	Summary of operation counts	33
	References	34

Abstract

An elementary derivation and a complete description are given of an algorithm for interpolation over a plane triangle when function values and first partial derivatives are given at the vertices. The method gives C^1 continuity with neighboring triangles.

The interpolation method is mathematically equivalent to one that has been discussed previously in the literature; however, the algorithmic form given here is more efficient than has previously been described.

C^1 - Compatible Interpolation Over A Triangle

1. Introduction

The problem treated in this report has been treated by numerous authors. See Birkhoff and Mansfield [1974] for extensive discussion of this and closely related problems and for other references. C^1 interpolation over triangular grids has application in structural analysis via finite element methods and in the computerized representation of surfaces for computer aided design.

The problem also arises in diverse scientific and engineering fields where it is useful to be able to construct a smooth surface that passes through a finite set of observed or computed values of some function $z = f(x, y)$. In these data-fitting applications, the desired end-product is often a contour plot of the interpolated function.

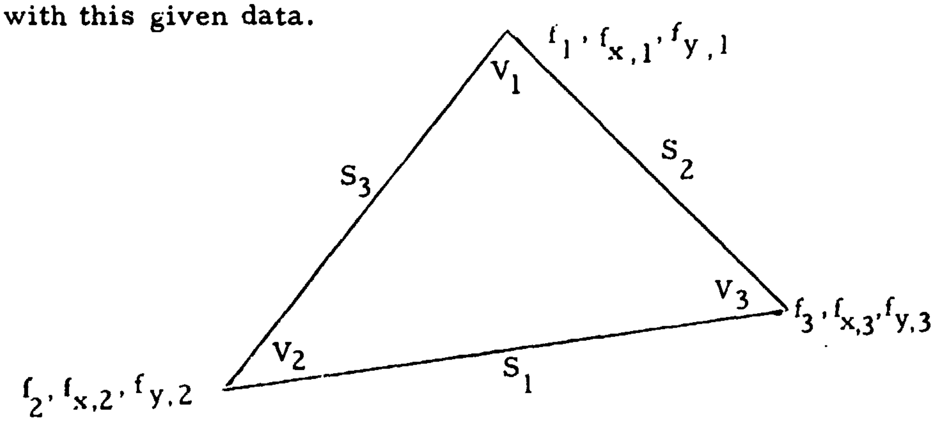
In this report, we give an elementary derivation and a complete algorithmic description of an interpolation method that is mathematically equivalent to one that is mentioned by Birkhoff and Mansfield, [1974], and specified in detail by J.-J. Goël, [1968]. Goël attributes the method to Clough and Tocher [1965] and Zienkiewicz [1967].

The algorithmic form in which the method is given here is more efficient for solving the interpolation problem than the form given by Goël [1968]. It is my present conjecture that one cannot expect to discover an algorithm for this problem that is significantly more efficient than the one given here. Other reports, yet to be written, will deal with the integration of this interpolation algorithm into a set of subprograms for constructing a triangular grid, Lawson [1972], and then doing look-up, C^1 interpolation, and contour plotting for a function, $z = f(x, y)$, whose values are given at a finite set of points.

The author thanks Dr. Fred T. Krogh for numerous fruitful discussions during the exploratory phase that preceded the writing of this paper and for a critical reading of the paper that produced numerous improvements.

2. The problem of C^1 - compatible interpolation over a triangle

Assume that values of a function, f, and its first partial derivatives, f_x and f_y, are given at the three vertices of a triangle, T, in the (xy)-plane. We wish to define a function w(x, y) for (x, y) in the triangle, T, that will agree with this given data.



With nine items of data being given we may anticipate that the interpolation method can be required to be exact for all polynomial functions of degree up to two, but not for all cubic functions, since the set of quadratic functions in two variables is a six-parameter family while the set of all cubics is a ten-parameter family. We will impose the requirement that the method be exact for quadratic polynomial data.

Furthermore we want the interpolation method to have the property that if it is applied to two adjacent triangles having an edge in common then function values and the first partial derivatives of the two interpolated functions will be identical along the common edge. Thus the method can be used for interpolation over a triangular grid, and the surface defined by the totality of the locally interpolated functions will have C^1 continuity over the entire region covered by the grid.

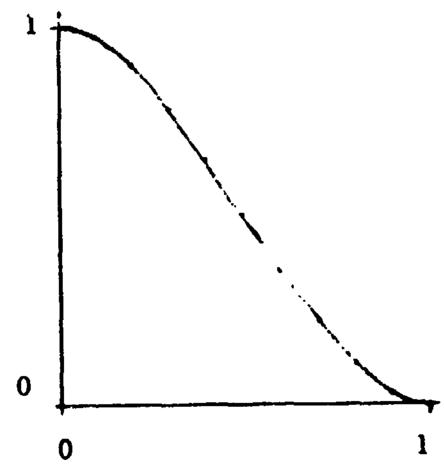
A convenient way of assuring that the interpolated functions on adjacent triangles have the same values and first partial derivatives along the common edge is to require that the values and first partial derivatives of the interpolated function along any edge must be determined only by the data given on that edge, i.e. the data given at the vertices at the ends of that edge.

Relative to a particular edge, say S_1, the given partial derivatives at vertices V_2 and V_3 can be rotated to give partial derivatives tangential to the edge and normal to the edge at V_2 and V_3. A fairly natural approach is to define values of w along side S_1 by Hermite cubic interpolation matching the required function values and tangential first partial derivative values at V_2 and V_3, and to define the first partial derivative of w normal to side S_1 to be a linear function along side S_1, taking the required values at V_2 and V_3.

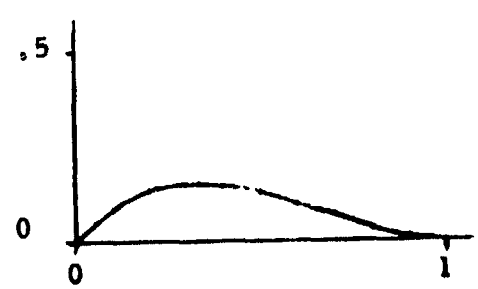
3. Some observations on one-dimensional Hermite cubic interpolation

Consider Hermite cubic interpolation on the unite interval $0 \le x \le 1$, with given data f_0 and f_0' at $x = 0$ and f_1 and f_1' at $x = 1$. The cardinal functions for this data are

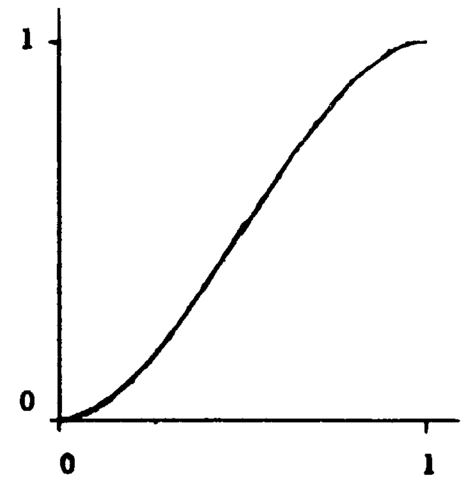
(1) $$\varphi_0(x) = (2x+1)(x-1)^2$$



(2) $$\overline{\varphi}_0(x) = x(x-1)^2$$

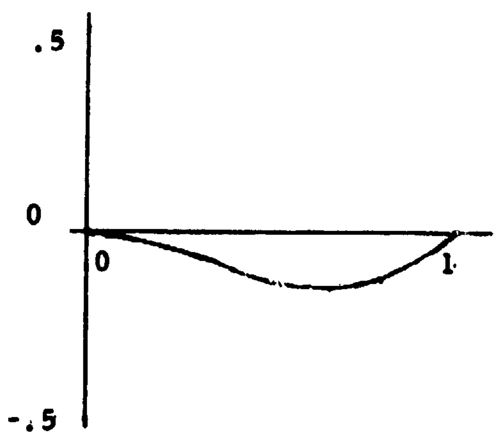


$$(3) \quad \varphi_1(x) = \varphi_0(1-x)$$



and

$$(4) \quad \overline{\varphi}_1(x) = -\overline{\varphi}_0(1-x)$$



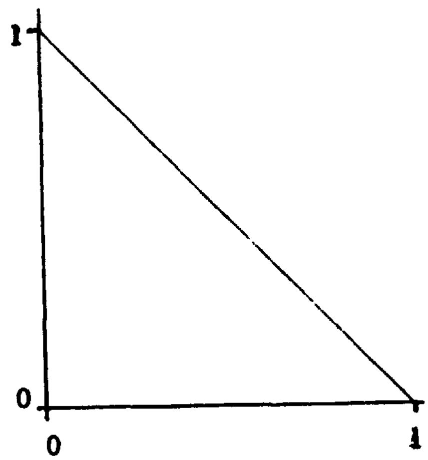
The interpolated cubic polynomial is given by

$$(5) \qquad w(x) = f_0\varphi_0(x) + f_0'\bar{\varphi}_0(x) + f\ \varphi_1(x) + f_1'\ \bar{\varphi}_1(x)$$

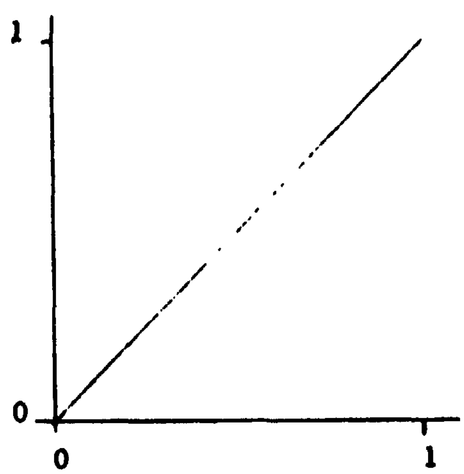
Note that all four of the cardinal functions in this formulation are cubic polynomials. In our interpolation problem over a triangle we shall find that cardinal functions of degree higher than two introduce relatively large increases in computational complexity. Thus it is useful to note that the solution to the one-dimensional Hermite cubic interpolation problem can be rearranged (in various ways) to involve at most one cubic basis function.

For example, we can use the four functions:

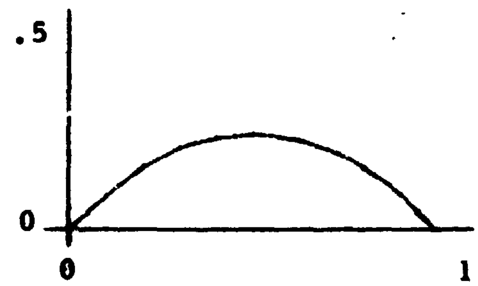
$$(6) \qquad \psi_1(x) = 1-x$$



$$(7) \qquad \hat{\psi}_1(x) = x$$



(8) $\psi_2(x) = x(1-x)$



and

(9) $\psi_3(x) = 2x(x-\frac{1}{2})(x-1)$

The same cubic polynomial as is defined by Eq (5) can then be constructed using the formulas

$$(10)\qquad m = f_1 - f_0$$

$$(11)\qquad w(x) = f_0\psi_1(x) + f_1\hat{\psi}_1(x) + \frac{f_0' - f_1'}{2}\psi_2(x) + \frac{f_1' + f_0' - 2m}{2}\psi_3(x)$$

Eq (11) is easily derived by constructing the formula in two stages. The first two terms clearly provide the linear interpolant that matches the data f_0 and f_1 at 0 and 1 respectively. This linear function has a slope of $m = f_1 - f_0$.

Thus, after subtracting this linear function, the remaining problem is to determine a cubic function having zero values at the endpoints and slopes of $f_0' - m$ at 0 and $f_1' - m$ at 1. Since ψ_2 has slopes of 1 and -1 at 0 and 1 respectively and ψ_3 has a slope of 1 at both 0 and 1 it follows that the function

$$(12)\qquad \frac{(f_0' - m) - (f_1' - m)}{2}\psi_2(x) + \frac{(f_0' - m) + f_1' - m)}{2}\psi_3(x)$$

will fit the residual data.

Combining this two-stage procedure into a single formula gives Eq (11).

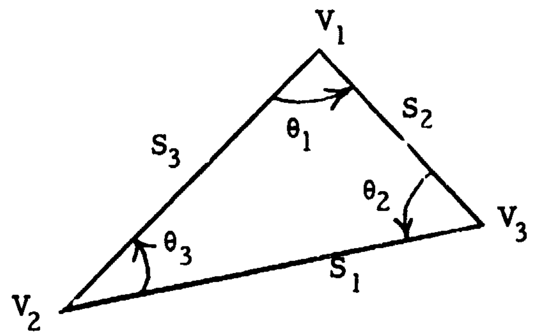
An analagous approach of combining a linear interpolant with a quadratic part and then a cubic part will be described in Section 5 for the interpolation problem over a triangle. Before commencing this derivation, however, we must introduce definitions and notation for the coordinate systems we will use over a triangle. This is the subject of Section 4.

4. Coordinate systems over plane triangles

Let T be a plane triangle having vertices V_1, V_2, and V_3. For convenience, we will think of these vertices as being labeled in counterclockwise order. Final results are not sensitive to this assumed ordering.

The indices in the formulas to follow take the values 1, 2, and 3. Index arithmetic is to be interpreted as being cyclic over these three values. For example, if $i = 3$, then $i + 1 = 1$ and $i + 2 = 2$.

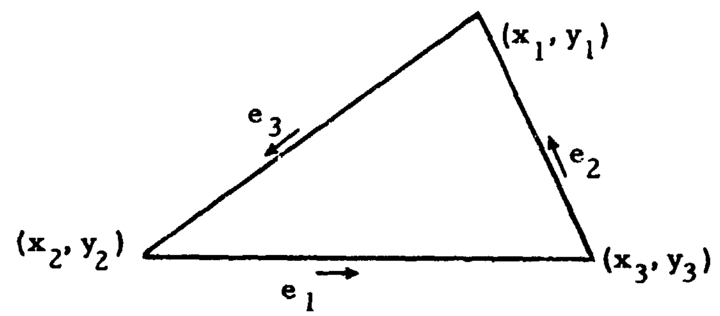
Let S_i denote the side of the triangle opposite to vertex V_i. Let θ_i denote the interior angle at vertex V_i, measured from side S_{i+2} to S_{i+1}



In a Euclidean (x, y)-plane let (x_i, y_i) be the coordinates of vertex V_i, $i = 1, 2, 3$.

Introduce directed edge vectors e_i with components u_i and v_i defined by

$$(13) \qquad e_i = \overline{V_{i+1}V_{i+2}} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} x_{i+2} - x_{i+1} \\ y_{i+2} - y_{i+1} \end{bmatrix}$$



Denote the Euclidean length of side S_i by

$$\ell_i = \|e_i\| = (u_i^2 + v_i^2)^{1/2} \qquad i = 1, 2, 3$$

Let c_i denote the inner product (dot product) of the two edge vectors directed away from vertex V_i, i.e. the inner product of e_{i-1} and $-e_{i+1}$.

(14) $$c_i = \mathrm{Dot}\,(\overline{V_i V_{i+1}}, \overline{V_i V_{i-1}}) = \mathrm{Dot}\,(e_{i-1}, -e_{i+1})$$

$$= -(u_{i-1}u_{i+1} + v_{i-1}v_{i+1})$$

$$= \ell_{i-1}\ell_{i+1} \cos \theta_i \qquad i = 1, 2, 3$$

We note in passing that the c_i's and ℓ_i's are related by the equations

(15) $$\ell_i^2 = c_{i-1} + c_{i+1} \qquad i = 1, 2, 3$$

(16) $$\ell_{i+1}^2 - \ell_{i-1}^2 = c_{i-1} - c_{i+1} \qquad i = 1, 2, 3$$

and

(17) $$2\, c_i = \ell_{i+1}^2 + \ell_{i-1}^2 - \ell_i^2 \qquad i = 1, 2, 3$$

Let δ denote twice the (signed) area of T. This quantity is representable as the scalar cross product of any two edge vectors directed away from a common vertex.

(18) $$\delta = \text{Cross}\,(\overline{V_iV_{i+1}},\ \overline{V_iV_{i-1}}) = \text{Cross}\,(e_{i-1},\ -e_{i+1})$$

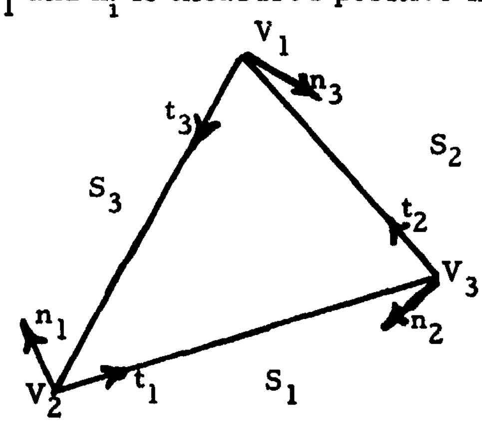
$$= \text{Cross}\,(e_{i+1},\ e_{i-1})$$

$$= \det \begin{bmatrix} u_{i+1} & u_{i-1} \\ v_{i+1} & v_{i-1} \end{bmatrix} = u_{i+1}v_{i-1} - v_{i+1}u_{i-1}$$

$$= \ell_{i-1}\ell_{i+1} \sin \theta_i \qquad i = 1, 2, 3$$

Tangential and Normal Coordinates

Relative to side S_i of the triangle, we introduce an orthogonal coordinate system using coordinates $(t_i,\ n_i)$ where t_i is measured along side S_i from V_{i+1} and n_i is measured positive in the inward normal direction.



The variables $(t_i,\ n_i)$ are related to the variables (x, y) by the equations

(19) $$\begin{bmatrix} t_i \\ n_i \end{bmatrix} = \frac{1}{\ell_i} \begin{bmatrix} u_i & v_i \\ -v_i & u_i \end{bmatrix} \cdot \begin{bmatrix} x-x_{i+1} \\ y-y_{i+1} \end{bmatrix} \qquad i = 1, 2, 3$$

and by the inverse equations

$$(20)\quad \begin{bmatrix} x-x_{i+1} \\ y-y_{i+1} \end{bmatrix} = \frac{1}{\ell_i} \begin{bmatrix} u_i & -v_i \\ v_i & u_i \end{bmatrix} \cdot \begin{bmatrix} t_i \\ n_i \end{bmatrix} \qquad i = 1, 2, 3$$

From Eq. (20) we obtain the partial derivatives

$$(21)\quad \left.\begin{aligned} \partial x/\partial t_i &= u_i/\ell_i \\ \partial x/\partial n_i &= -v_i/\ell_i \\ \partial y/\partial t_i &= v_i/\ell_i \\ \partial y/\partial n_i &= u_i/\ell_i \end{aligned}\right\} \qquad i = 1,\ 2,\ 3$$

<u>Barycentric or Areal Coordinates</u>

Let P be a point with coordinates (x, y). Define the three barycentric (or areal) coordinates of P by:

$$(22)\quad \begin{aligned} r_j &= \mathrm{Cross}\,(\overline{V_{j+1}V_{j+2}},\ \overline{V_{j+1}P})\ /\ \delta \\ &= \mathrm{Cross}\,(e_j,\ \overline{V_{j+1}P})\ /\ \delta \\ &= \delta^{-1}\det \begin{bmatrix} u_j & x-x_{j+1} \\ v_j & y-y_{j+1} \end{bmatrix} \\ &= \delta^{-1}\left[u_j(y-y_{j+1}) - v_j(x-x_{j+1})\right] \qquad j = 1, 2, 3 \end{aligned}$$

Note that the quantity computed by the cross product in the formula for r_j is twice the area of the triangle formed by side j and the point P. Thus the sum of the three cross products used to compute r_1, r_2, and r_3 must be twice the area of T. Therefore, with the normalization factor δ^{-1} appearing in the formulas it follows that

(23) $$r_1 + r_2 + r_3 = 1$$

The barycentric coordinates are the unique set of numbers having unit sum and representing P as a linear combination of $V_1 V_2$, and V_3, thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{j=1}^{3} r_j \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

Each barycentric coordinate r_j is the unique linear function of (x, y) that is zero along the line determined by side S_j and takes the value 1 at the vertex V_j.

For points inside the triangle T we have all $r_j \geq 0$ while for points outside there will be some $r_j < 0$. The barycentric coordinates of the vertices are:

$$V_1 \sim (1, 0, 0)$$

$$V_2 \sim (0, 1, 0)$$

$$V_3 \sim (0, 0, 1)$$

Using Eq. (22) we may compute partial derivatives as follows:

(24) $$\begin{aligned} \partial r_j / \partial x &= -v_j / \delta \\ \partial r_j / \partial y &= u_j / \delta \end{aligned} \qquad j = 1, 2, 3$$

Using Eq. (21) and (24) we obtain further partial derivatives:

(25) $$\begin{aligned} \partial r_j / \partial t_i &= (\partial r_j / \partial x)(\partial x / \partial t_i) + (\partial r_j / \partial y)(\partial y / \partial t_i) \\ &= (-v_j u_i + u_j v_i) \,/\, (\ell_i \delta) \\ &= \text{Cross}\,(e_j,\ e_i) \,/\, (\ell_i \delta) \end{aligned}$$

$$= \begin{cases} 0 & \text{if } j = i \\ 1/\ell_i & \text{if } i-j = 1 \\ -1/\ell_i & \text{if } j-i = 1 \end{cases}$$

(26) $$\partial r_j/\partial n_i = (\partial r_j/\partial x)(\partial x/\partial n_i) + (\partial r_j/\partial y)(\partial y/\partial n_i)$$

$$= (v_j v_i + u_j u_i) / (\ell_i \delta)$$

$$= \mathrm{Dot}\,(e_j, e_i) / (\ell_i \delta)$$

$$= \begin{cases} \ell_i^2/(\ell_i \delta) & \text{if } i = j \\ -c_{i-1}/(\ell_i \delta) & \text{if } j-i = 1 \\ -c_{i+1}/(\ell_i \delta) & \text{if } i-j = 1 \end{cases}$$

For more convenient reference, we organize the results of Eq. (25) and (26) into tables as follows:

Table 1 . Values of $\partial r_j/\partial t_i$

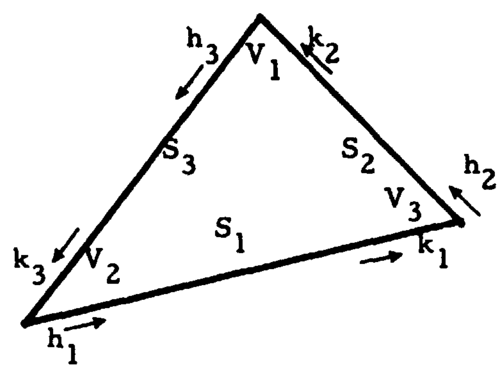
	$/\partial t_1$	$/\partial t_2$	$/\partial t_3$
$\partial r_1/$	0	$1/\ell_2$	$-1/\ell_3$
$\partial r_2/$	$-1/\ell_1$	0	$1/\ell_3$
$\partial r_3/$	$1/\ell_1$	$-1/\ell_2$	0

Table 2 . Values of $\partial r_j/\partial n_i$

	$/\partial n_1$	$/\partial n_2$	$/\partial n_3$
$\partial r_1/$	$\ell_1^2/(\ell_1\delta)$	$-c_3/(\ell_2\delta)$	$-c_2/(\ell_3\delta)$
$\partial r_2/$	$-c_3/(\ell_1\delta)$	$\ell_2^2/(\ell_2\delta)$	$-c_1/(\ell_3\delta)$
$\partial r_3/$	$-c_2/(\ell_1\delta)$	$-c_1/(\ell_2\delta)$	$\ell_3^2/(\ell_3\delta)$

5. Constructing a solution

First convert the given partial derivative data at each vertex to partial derivatives in the directions of the edges meeting at the vertex. For i = 1, 2, and 3, let h_i and k_i respectively denote the values of the first partial derivative with respect to t_i (the tangential direction along side S_i) at $t_i = 0$ and $t_i = 1$ (i.e., at the vertices V_{i+1} and V_{i-1}).



The values of h_i and k_i are given by

(27)
$$\left.\begin{aligned} h_i &= (u_i f_{x,\, i+1} + v_i f_{y,\, i+1})/\ell_i \\ k_i &= (u_i f_{x,\, i-1} + v_i f_{y,\, i-1})/\ell_i \end{aligned}\right\} i = 1,\ 2,\ 3$$

Henceforth, we may regard f_i, h_i, and k_i for i = 1, 2, 3, as defining the data to be interpolated. We proceed by analogy with the discussion of the one-dimensional problem in Section 3 . A linear function interpolating the function values f_i, i = 1, 2, 3, is given by

$$\omega^{(1)} = r_1 f_1 + r_2 f_2 + r_3 f_3$$

Along side S_i, this function has slope

(28)
$$m_i = (f_{i-1} - f_{i+1}) / \ell_i$$

with respect to the tangential variable t_i. Subtracting the vertex values and partial derivatives of $\omega^{(1)}$ from the given data, we are left with the residual problem of interpolating vertex values of zero and tangential partial derivative values of $h_i - m_i$ and $k_i - m_i$, $i = 1, 2, 3$.

Let us temporarily restrict attention to one side, say side S_1. On side S_1 we have $r_1 = 0$, $r_2 = 1 - r_3$, and $t_1 = \ell_1 r_3$. Note that the quadratic function $r_2 r_3$ reduces to $r_3(1-r_3)$, and the cubic function $r_2 r_3(r_2 - r_3)$ reduces to $2r_3(r_3 - \frac{1}{2})(r_3 - 1)$ along side S_1. [Compare with ψ_2 and ψ_3 of Section 3 .] It is easily verified that the partial derivative with respect to t_1 of the scaled function $\ell_1 r_2 r_3$ has the values 1 and -1 at the vertices V_2 and V_3 respectively. Similarly $\partial[\ell_1 r_2 r_3(r_2 - r_3)]/\partial t_1$ has the value 1 at V_2 and V_3.

Thus the function

$$(29) \qquad \omega_1^{(2)} = \frac{h_1 - k_1}{2} \ell_1 r_2 r_3 + \frac{h_1 + k_1 - 2m_1}{2} \ell_1 r_2 r_3 (r_2 - r_3)$$

satisfies the residual interpolation requirements on side S_1; i.e., $\omega_1^{(2)}$ has zero values at V_2 and V_3 and its partial derivative with respect to t_1 has the values $h_1 - m_1$, and $k_1 - m_1$ at V_2 and V_3 respectively.

On side S_2 we have $r_2 = 0$, and thus $\omega_1^{(2)}$ and its tangential partial derivative are zero there. Similarly, since $r_3 = 0$ on side S_3, $\omega_1^{(2)}$ and its tangential partial derivative are also zero on side S_3.

By appropriate cycling of indices, define functions $\omega_2^{(2)}$ and $\omega_3^{(2)}$ analagous to $\omega_1^{(2)}$:

$$(30) \qquad \omega_i^{(2)} = \frac{h_i - k_i}{2} \ell_i r_{i+1} r_{i-1} + \frac{h_i + k_i - 2m_i}{2} \ell_i r_{i+1} r_{i-1} (r_{i+1} - r_{i-1})$$

for $i = 1, 2, 3$

It follows that the function

$$(31)\qquad \bar{\omega} = \omega^{(1)} + \sum_{i=1}^{3} \omega_i^{(2)}$$

interpolates the nine items of data f_i, h_i, k_i, $i=1, 2, 3$.

Note further that $\bar{\omega}$ is exact for quadratic functions since if the data f_i, h_i, k_i, $i=1, 2, 3$, arise from a quadratic function it will follow that

$$h_i - m_i = -(k_i - m_i) \qquad i = 1, 2, 3$$

so the coefficients of the cubic terms in $\bar{\omega}$ vanish leaving $\bar{\omega}$ as the unique quadratic function matching the given data.

The function $\bar{\omega}$ has a defect however. We require that the partial derivative normal to any side must be a linear function along that side. The cubic functions $r_{i+1} r_{i-1}(r_{i+1} - r_{i-1})$, $i=1, 2, 3$, do not have this property and thus, in general, $\bar{\omega}$ does not.

The remedy, described in Goël [1968], is to introduce correction functions ρ_i, $i=1, 2, 3$. The function ρ_i is required to be zero on all three sides of the triangle. It follows that its first partial derivatives in all directions at each vertex are zero. It is further required that the normal derivatives of ρ_i relative to sides S_{i+1} and S_{i-1} be zero on those sides respectively, while the normal derivative of ρ_i relative to side S_i is to be a quadratic function along that side. Specifically we can require that

$$(32)\qquad \partial\rho_i/\partial n_i \Big|_{\text{on side } S_i} = r_{i-1}(1 - r_{i-1})\, \ell_i/6$$

By adding appropriate multiples of ρ_1, ρ_2, and ρ_3 to a cubic function, such as $r_2 r_3(r_2 - r_3)$, one can construct a function whose normal partial derivatives on each side are linear functions along the respective sides.

This point will be further developed in the next three sections. In Section 6 we determine the multiples of ρ_1, ρ_2, and ρ_3 needed to correct the function $r_2r_3(r_2-r_3)$. In Sections 7 and 8, we discuss two distinct sets of functions having the properties required of the ρ_i's.

6. Correcting the normal derivatives of $g_1 = r_2 r_3(r_2 - r_3)$

Define

$$g_1 = r_2 r_3 (r_2 - r_3)$$

The partial derivatives of g_1 with respect to the r_i's are

$$\partial g_1/\partial r_1 = 0$$

$$\partial g_1/\partial r_2 = 2r_2 r_3 - r_3^2$$

$$\partial g_1/\partial r_3 = r_2^2 - 2r_2 r_3$$

Using the expressions for $\partial r_i/\partial n_j$ given in Table 2, and evaluating on the indicated sides we obtain:

$$(33) \qquad \partial g_1/\partial n_1 \Big|_{\text{on side } S_1} = [3(c_3 - c_2) r_3^2 + 2(2c_2 - c_3) r_3 - c_2]/(\ell_1 \delta)$$

$$(34) \qquad \partial g_1/\partial n_2 \Big|_{\text{on side } S_2} = -\ell_2 (1 - r_1)^2/\delta$$

$$(35) \qquad \partial g_1/\partial n_3 \Big|_{\text{on side } S_3} = \ell_3 r_2^2/\delta$$

We assume the availablity of C^1 functions ρ_i, i=1, 2, 3, which have zero values along all edges and satisfy

$$(36) \qquad \partial \rho_i/\partial n_j \Big|_{\text{on side } S_i} = \begin{cases} r_{i-1}(1 - r_{i-1})\ell_i/\delta & \text{if } j=i \\ 0 \text{ if } j=i-1 \text{ or } j=i+1 \end{cases}$$

We wish to determine coefficient α_{11}, α_{12}, and α_{13} such that the function

$$\tilde{g}_1 = g_1 + \alpha_{11}\rho_1 + \alpha_{12}\rho_2 + \alpha_{13}\rho_3$$

will have the property that for $j = 1, 2, 3$, the normal derivative $\partial\tilde{g}_1/\partial n_j$ is a linear function along side S_j. Comparing the quadratic terms in Eqs. (33), (34), and (35) with the quadratic term in Eq. (36) it follows that the appropriate values for the α_{1j}'s are

$$\alpha_{11} = 3(c_3 - c_2)/\ell_1^2 = 3(\ell_2^2 - \ell_3^2)/\ell_1^2$$

$$\alpha_{12} = -1$$

and

$$\alpha_{13} = 1$$

Using these values of the α_{1j}'s one can obtain the following equations showing the linear character of the normal derivatives of $\ell_1\tilde{g}_1$ on the respective sides:

$$\partial(\ell_1\tilde{g}_1)/\partial n_1 \Big|_{\text{on side } S_1} = [(c_3+c_2)r_3 - c_2]/\delta$$

$$= (c_3 r_3 - c_2 r_2)/\delta$$

$$= \frac{r_3}{\tan\theta_3} - \frac{r_2}{\tan\theta_2}$$

$$\partial(\ell_1\tilde{g}_1)/\partial n_2 \Big|_{\text{on side } S_2} = -\ell_1\ell_2(1-r_1)/\delta = -\ell_1\ell_2 r_3/\delta$$

$$= -r_3/\sin\theta_3$$

$$\partial(\ell_1\tilde{g}_1)/\partial n_3 \Big|_{\text{on side } S_3} = \ell_1\ell_3 r_2/\delta = r_2/\sin\theta_2$$

Collecting the results of this section and cycling the indices appropriately we define

$$(37)\qquad \tilde{g}_i = r_{i+1}r_{i-1}(r_{i+1}-r_{i-1}) + \frac{3(\ell_{i+1}^2 - \ell_{i-1}^2)}{\ell_i^2}\rho_i-\rho_{i+1}+\rho_{i-1}$$

$$i = 1,\ 2,\ 3$$

Each function $\tilde{g}_i$ has the same interpolatory properties as the simpler cubic function $r_{i+1}r_{i-1}(r_{i+1}-r_{i-1})$ at the three vertices and has the additional property that $\partial\tilde{g}_i/\partial n_j$ is a linear function along side S_j for all i and j.

Therefore for our complete interpolation formula we replace Eq (31) by

$$(38)\qquad w = \sum_{i=1}^{3}\left[r_i f_i + \frac{h_i-k_i}{2}\ell_i r_{i-1}r_{i+1} + \frac{h_i+k_i-2m_i}{2}\ell_i\tilde{g}_i\right]$$

7. A set of rational correction functions, ρ_i

Define

$$\rho_i = r_i r_{i+1}^2 r_{i-1}^2/[(1-r_{i+1})(1-r_{i-1})] \qquad i = 1, 2, 3 \tag{39}$$

This set of rational functions is discussed by Goël [1968] who attributes their use in this context to Zienkiewicz [1967].

For convenience consider the single function

$$\rho_1 = r_1 r_2^2 r_3^2 \;/\; [(1-r_2)(1-r_3)]$$

Over the triangle, T, the denominator of ρ_1 vanishes only at the vertex V_2, where $r_2 = 1$, and at V_3, where $r_3 = 1$. These are removable singularities however since, for example, at V_2 the numerator has a third order zero ($r_1 = 0$ and $r_3 = 0$) and thus ρ_1 has a second order zero at V_2. Similarly ρ_1 vanishes to second order at V_3.

At all other edge points it is clear that ρ_1 vanishes because it contains r_1, r_2, and r_3 as factors.

To verify that the normal partial derivatives of ρ_1 have the necessary properties compute

$$\partial\rho_1/\partial r_1 = \rho_1/r_1$$

$$\partial\rho_1/\partial r_2 = (2-r_2)\rho_1/[r_2(1-r_2)]$$

$$\partial\rho_1/\partial r_3 = (2-r_3)\rho_1/[r_3(1-r_3)]$$

Then using expressions for $\partial r_i/\partial n_j$ from Table 2, one finds

$$\partial\rho_1/\partial n_1\Big|_{\text{on side } S_1} = r_3(1-r_3)\ell_1/\delta$$

$$\partial\rho_1/\partial n_2\Big|_{\text{on side } S_2} = 0$$

$$\partial\rho_1/\partial n_3\Big|_{\text{on side } S_3} = 0$$

It is thus verified that the rational functions of Eq(39) can be used as the correction functions ρ_i described in Section 5.

To compute values of ρ_i, i = 1, 2, 3, assume values of r_i, i = 1, 2, 3, and

(40) $$\varphi_i = r_{i+1}r_{i-1} \qquad i = 1, 2, 3$$

are given. One then computes

$$\psi := r_1\varphi_1$$

$$\hat{r}_i := 1-r_i \qquad i = 1, 2, 3$$

{If any $\hat{r}_i = 0$ branch to handle the special trivial case of interpolation at a vertex}

$$\rho_i := \psi\varphi_i/(\hat{r}_{i+1}\hat{r}_{i-1}) \qquad i = 1,2,3$$

Thus the computation of the rational ρ_i's requires 7 multiplications, 3 additions, 3 divisions, and 3 zero tests.

8. A set of piecewise cubic correction functions, ρ_i

Define

$$(41)\quad \rho_i = \begin{cases} r_i[6r_{i+1}r_{i-1} + r_i(5r_i - 3)]/6 & \text{if } r_i = \min\{r_1, r_2, r_3\} \\ r_{i+1}^2(-r_{i+1} + 3r_{i-1})/6 & \text{if } r_{i+1} = \min\{r_1, r_2, r_3\} \\ r_{i-1}^2(-r_{i-1} + 3r_{i+1})/6 & \text{if } r_{i-1} = \min\{r_1, r_2, r_3\} \end{cases}$$

for $i = 1, 2, 3$

Each function ρ_i consists of a set of three cubic functions which match with C^1 continuity along internal boundary lines connecting the vertices to the centroid of the triangle T. This set of functions is discussed by Goël [1968] who attributes their use in this context to Clough and Tocher [1965].

Along side S_1 we have $r_1 = \min\{r_1, r_2, r_3\}$ and thus

$$\rho_1 = r_1[6r_2r_3 + r_1(5r_1 - 3)]/6$$

In this region the derivatives $\partial\rho_1/\partial r_i$ are given by

$$\partial\rho_1/\partial r_1 = [6r_2r_3 + 15r_1^2 - 6r_1]/6$$

$$\partial\rho_1/\partial r_2 = r_1r_3$$

$$\partial\rho_1/\partial r_3 = r_1r_2$$

Using the expressions for $\partial r_i/\partial n_1$ from Table 2 , we find

$$\partial\rho_1/\partial n_1 \Big|_{\text{on side } S_1} = r_3(1-r_3)\ell_1/\delta$$

It can also be verified that

$$\partial\rho_1/\partial n_2 \Big|_{\text{on side } S_2} = 0$$

and

$$\partial\rho_1/\partial n_3 \Big|_{\text{on side } S_3} = 0$$

Thus the piecewise cubic functions of Eq (41) have the properties need for use as the correction functions ρ_i of Section 5.

If Eq (41) is used in a computer program the usual situation will be that for one set of values of r_1, r_2, and r_3 the program must compute values of ρ_1, ρ_2, and ρ_3. In this context the following restatement of Eq(41) is useful:

Let m be an index such that $r_m = \min\{r_1, r_2, r_3\}$. Then

$$\rho_m = r_m[6r_{m+1}r_{m-1} + r_m(5r_m-3)]/6$$

$$\rho_{m+1} = r_m^2(-r_m+3r_{m-1})/6$$

$$\rho_{m-1} = r_m^2(-r_m+3r_{m+1})/6$$

To compute values of ρ_i, i=1, 2, 3, given r_i, i=1, 2, 3, and

$$\varphi_i = r_{i+1}r_{i-1} \qquad i = 1, 2, 3$$

the following steps can be used:

Find m such that $r_m = \min\{r_1, r_2, r_3\}$

$$a := \frac{1}{2} r_m^2$$

$$b := \frac{1}{3} r_m$$

$$\rho_m := r_m(\varphi_m + \frac{5}{3} a) - a$$

$$\rho_{m+1} := a(r_{m-1} - b)$$

$$\rho_{m-1} := a(r_{m+1} - b)$$

Two compares are required to determine m. Counting these as additions, the computation of the piecewise cubic correction functions, ρ_i, requires seven multiplications and six additions.

9. Transforming formulas to algorithms

Assume that data $(x_i, y_i, f_i, f_{x,i}, f_{y,i})$, $i = 1, 2, 3$ is given, where (x_i, y_i) specifies the coordinates of the vertex V_i and $(f_i, f_{x,i}, f_{y,i})$ specifies the value of the function and its first partial derivatives with respect to x and y at the vertex V_i. The vertices V_i should not be colinear. Either counterclockwise or clockwise ordering of the vertices is acceptable.

Further assume a coordinate pair, (x, y), is given at which an interpolated value, w, is to be computed. The point (x, y) should not be exterior to the triangle T having vertices V_1, V_2, and V_3.

We will describe the interpolation algorithm in three phases. Phase 1 will compute the preliminary quantities, u_i, v_i, ℓ_i^2, δ^{-1}, r_i, and φ_i, $i = 1, 2, 3$. Phase 2 will be the computation of the ρ_i's using either the method of Section 7 or of Section 8 . Phase 3 will complete the interpolation. We will describe three versions of Phase 3.

All index expressions such as i+1 and i-1 must be interpreted cyclicly so that the resulting index value is 1, 2, or 3. The name "det" is used to indicate computation of the determinant of a matrix. Phase 1 of the algorithm proceeds as follows:

Begin Phase 1

$$\left.\begin{aligned} u_i &:= x_{i-1} - x_{i+1} \\ v_i &:= y_{i-1} - y_{i+1} \\ \ell_i^2 &:= u_i^2 + v_i^2 \end{aligned}\right\} \quad i = 1, 2, 3$$

$$\delta := \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

{Test for the error condition, $\delta=0$, indicating colinearity of the of the vertices.}

$$\tilde{x} := x - x_1$$

$$\tilde{y} := y - y_1$$

$$r_2 := \delta^{-1} \det \begin{bmatrix} u_2 & \tilde{x} \\ v_2 & \tilde{y} \end{bmatrix}$$

$$r_3 := \delta^{-1} \det \begin{bmatrix} u_3 & \tilde{x} \\ v_3 & \tilde{y} \end{bmatrix}$$

$$r_1 := 1-(r_2 + r_3)$$

{If one wishes to test for the possibility that (x, y) is exterior to the triangle the test can be made here. The condition $r_i < 0$ for any i indicates that (x, y) is exterior to the triangle.}

$$\varphi_i := r_{i+1} r_{i-1} \quad i = 1, 2, 3$$

End Phase 1

Note that Phase 1 requires 17 multiplications, 16 additions, and one division.

Phase 2 consists of the computation of the ρ_i's using either the method of Section 7 or of Section 8. We proceed to the discussion of Phase 3.

The divisor ℓ_i in Eq.(27), and (28) will cancel with the multiplier ℓ_i in Eq (38). Thus instead of computing h_i and k_i we will compute quantities $\tilde{h}_i$ and $\tilde{k}_i$ such that $h_i = \tilde{h}_i / \ell_i$ and $k_i = \tilde{k}_i / \ell_i$.

Version 1 of Phase 3 is the following:

Phase 3, Version 1

$$\left.\begin{aligned}
\tilde{h}_i &:= u_i f_{x,i+1} + v_i f_{y,i+1} \\
\tilde{k}_i &:= u_i f_{x,i-1} + v_i f_{y,i-1} \\
\tilde{g}_i &:= (r_{i+1} - r_{i-1})\varphi_i + 3\,\frac{\ell_{i+1}^2 - \ell_{i-1}^2}{\ell_i^2}\,\rho_i - \rho_{i+1} + \rho_{i-1}
\end{aligned}\right\} \quad i = 1, 2, 3$$

$$w := \sum_{i=1}^{3} f_i r_i + \frac{1}{2}(\tilde{h}_i - \tilde{k}_i)\,\varphi_i + [\tfrac{1}{2}(\tilde{h}_i + \tilde{k}_i) - f_{i-1} + f_{i+1}]\,\tilde{g}_i$$

Phase 3, Version 1, requires 36 multiplications, 42 additions, and 3 divisions. This includes six multiplications by one half. The computation can be re-arranged so that there is only one multiplication by one half. This leads to what we will call Phase 3, Version 2 which requires 31 multiplications, 43 additions, and 3 divisions.

Version 2 differs from Version 1 only in the expression for w which changes to:

$$w := \sum_{i=i}^{3} f_i(r_i + \tilde{g}_{i-1} - \tilde{g}_{i+1}) + \frac{1}{2}\sum_{i=1}^{3} \tilde{h}_i\,(\tilde{g}_i + \varphi_i) + \tilde{k}_i(\tilde{g}_i - \varphi_i)$$

From this point it takes only a little more rearrangement to obtain a formulation which explicitly uses cardinal functions for the given data $(f_i, f_{x,i}, f_{y,i})$, $i = 1, 2, 3$ This formulation, which we will call Phase 3, Version 3, uses the same number of multiplications, additions, and divisions as Version 2.

Phase 3, Version 3

$$\left.\begin{aligned}
\tilde{g}_i &:= (r_{i+1} - r_{i-1})\varphi_i + 3\,\frac{\ell_{i+1}^2 - \ell_{i-1}^2}{\ell_i^2}\,\rho_i - \rho_{i+1} + \rho_{i-1} \\
p_i &:= \tilde{g}_i + \varphi_i \\
q_i &:= \tilde{g}_i - \varphi_i
\end{aligned}\right\} \quad i = 1,\ 2,\ 3$$

$$\left.\begin{aligned}
\alpha_i &:= r_i + \tilde{g}_{i-1} - \tilde{g}_{i+1} \\
\tilde{\beta}_i &:= u_{i-1}p_{i-1} + u_{i+1}q_{i+1} \\
\tilde{\gamma}_i &:= v_{i-1}p_{i-1} + v_{i+1}q_{i+1}
\end{aligned}\right\} \quad i = 1,\ 2,\ 3$$

$$w := \sum_{i=1}^{3} f_i\alpha_i + \frac{1}{2}\sum_{i=1}^{3}\left(f_{x,i}\,\tilde{\beta}_i + f_{y,i}\,\tilde{\gamma}_i\right)$$

Version 3 is particularly efficient for the case in which there are a number of functions to be interpolated at the same interpolation point, (x, y). In such a case only the final formula for w must be recomputed for each function to be interpolated.

The explicit computation of the cardinal functions, α_i, $\frac{1}{2}\tilde{\beta}_i$, and $\frac{1}{2}\tilde{\gamma}_i$, as provided by Version 3, is needed in applications in which the quantities f_i, $f_{x,i}$, and $f_{y,i}$ are unknowns to be solved for. This is the situation in the finite element methods for solving partial differential equations and in the fitting of a smooth surface to noisy data.

10. The method described in Goël [1968]

For comparison with the results of Section 9 we will give a brief description of the interpolation method given in Goël [1968]. This method is attributed by Goël to Clough and Tocher [1965] and Zienkiewicz [1967].

We refer primarily to Eq (28) and (34) of Goël [1968]. The following change of notation will convert Goëls symbols to ours.

Goël's notation	Notation of this paper
x'	r_2
y'	r_3
$1-x'-y'$	r_1
ρ_i	$-\rho_i/6$
$\alpha_i, \beta_i, \gamma_i$	$\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i$
$\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$	$\alpha_i, \beta_i, \gamma_i$
Δ	$\delta/2$
L_i, C_i	ℓ_i, c_i

We will indicate precomputation of the common subexpressions in Goël's formulas in order to provide a basis for obtaining a realistic operation count. The quantities computed in Phases 1 and 2 [see Section 9] are all needed for Goël's formulas so we will assume these computations have been done and proceed to describe Phase 3.

Phase 3, Goël

$$
\begin{aligned}
\eta_i &:= r_i^2\,(2r_{i+1} - 3) \\
\tilde{\rho}_i &:= \rho_i / \ell_i^2 \\
\psi &:= r_1 \varphi_1
\end{aligned}
$$

$$
\left.
\begin{aligned}
\hat{\alpha}_i &:= 1 + \eta_{i+1} + \eta_{i-1} - 2\,[\rho_i - 2\,(\rho_{i+1} + \rho_{i-1} - \psi)] \\
\hat{\beta}_i &:= r_i \varphi_{i-1} + \frac{1}{2}\,[\psi - \rho_i - 5\rho_{i+1} + 3\rho_{i-1}] \\
\hat{\gamma}_i &:= r_i \varphi_{i+1} + \frac{1}{2}\,[\psi - \rho_i + 3\rho_{i+1} - 5\rho_{i-1}]
\end{aligned}
\right\}\ i = 1,\ 2,\ 3
$$

$$
\left.
\begin{aligned}
\alpha_i &:= \hat{\alpha}_i - (\rho_{i+1} + \rho_{i-1})(u_{i+1}u_{i-1} + v_{i+1}v_{i-1}) \\
\beta_i &:= u_{i-1}\hat{\beta}_i - u_{i+1}\,\hat{\gamma}_i + \frac{1}{2}\,\delta\,(v_{i+1}\tilde{\rho}_{i+1} + v_{i-1}\tilde{\rho}_{i-1}) \\
\gamma_i &:= v_{i-1}\hat{\beta}_i - v_{i-1}\hat{\gamma}_i - \frac{1}{2}\,\delta\,(u_{i+1}\tilde{\rho}_{i+1} + u_{i-1}\tilde{\rho}_{i-1})
\end{aligned}
\right\}\ i = 1,\ 2,\ 3
$$

$$
w := \sum_{i=1}^{3} (f_i\,\alpha_i + f_{x,i}\,\beta_i + f_{y,i}\,\gamma_i)
$$

Assuming the quantities $\frac{1}{2}\delta$, $3\rho_i$, and $5\rho_i$, would be computed only once each, the operation count for Phase 3, Goël, is 71 multiplications, 81 additions, and 3 divisions.

It can be verified, by the appropriate tedious algebra, that α_i, β_i, and γ_i as defined above are identical with α_i, $\frac{1}{2}\tilde{\beta}_i$, and $\frac{1}{2}\tilde{\gamma}_i$ as defined in Phase 3, Version 3 [Section 9]. Thus the interpolated value, w, is the same by either method.

11. Summary of operation counts

	Multiplications	Additions	Divisions
Phase 1	17	16	1
Phase 2 using rational ρ_i's	7	3	3
Phase 2 using piecewise cubic ρ_i's	7	6	0
Phase 3, Version 1	36	42	3
Phase 3, Versions 2 or 3	31	43	3
Phase 3, Goël	71	81	3
Totals using piecewise cubic ρ_i's:			
Version 1	60	64	4
Versions 2 or 3	55	65	4
Goël	95	103	4

If we weight the multiplications, additions, and divisions in the ratio 2:1:6, then the operation count for Versions 2 or 3 is 63% of the count for Goël's version.

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